Modeling Student Understanding of Foundational Concepts Related to Slope:

An Application of the Attribute Hierarchy Method

Angela Broaddus, Ph.D.
Formative and Interim Assessment Programs Coordinator
Center for Educational Testing and Evaluation
1122 West Campus Road
JR Pearson Hall, Room 748
Lawrence, KS 66045
785-864-2916
broaddus@ku.edu

Author note

Angela Broaddus, Center for Educational Testing and Evaluation, University of Kansas.
The author received no financial support for the research or authorship of this article.
Correspondence concerning this article should be addressed to Angela Broaddus, Center for Educational Testing and Evaluation, University of Kansas, 1122 West Campus Road, JR Pearson Hall, Room 748, Lawrence, KS 66045; E-mail: broaddus@ku.edu.
Abstract

The purpose of this study was to propose a cognitive model hypothesizing how conceptual understanding of slope develops, to design an instrument to assess understanding of selected foundational concepts related to slope as described in the model, and to determine the model’s accuracy by analyzing student test data. The instrument was administered to Kansas students in middle and high school mathematics courses. Test responses were analyzed using item response theory, and students were classified into knowledge states based on their responses using the attribute hierarchy method. The findings suggest that knowledge develops in a manner consistent with the proposed cognitive model, supporting the theory that there are three main levels of understanding of the selected foundational concepts related to slope.
Educational tests are implemented in schools to gather evidence that can inform a variety of decisions about students. In many cases, student test scores are interpreted as measures of what students know about a particular knowledge domain. However, these scores do not always provide the detailed information that students and teachers need to correct misunderstandings and improve achievement. Construct-centered tests that originate from descriptions of the knowledge, skills, and abilities students are expected to demonstrate offer the potential to provide practitioners with detailed test data that can inform valid instructional decisions (Haertel, 1985; Mislevy & Haertel, 2006). Such instruments can provide students and teachers with feedback that describes student performance in terms of the learning targets assessed instead of merely test scores.

Unfortunately, few tests are designed specifically to reflect theories about how particular content is learned and cognitively organized (Pellegrino, 1988; Pellegrino, Baxter, & Glaser, 1999). Cognitive models offer test designers information about what it means to understand a particular body of content knowledge, which should in turn influence the nature of assessments that target that particular body of knowledge. Mislevy and Haertel (2006) urged test developers to incorporate cognitive models in order to establish clear links between assessment design and proposed interpretations of test scores.

This study is an example of how educational literature can be used to guide the design of both a proposed theoretical cognitive model and a corresponding assessment
and demonstrates how student test responses can be used to reflect on the validity of that model. This is the process described by Mislevy and Haertel (2006) and advocated by Pellegrino (1988) and Pellegrino, Baxter, and Glaser (1999), but it has rarely been used to guide the development of an actual assessment.

**Background**

Learning with understanding is important (Novak, 1998) and depends on the human ability to meaningfully relate new information to what is already known and cognitively structured (Ausubel, 1968). Learning with understanding occurs when students actively and consciously connect a new idea to their existing cognitive structure (Ausubel, 1968). Concepts that are learned with understanding are more easily remembered and applied to future experiences than concepts that are not purposefully connected to a person’s existing knowledge (Grouws, 1991). Therefore, learning with understanding provides the foundation for future learning.

**Understanding Mathematics**

Educators and researchers continue to emphasize the importance of making sense of mathematics (CCSSO/NGA, 2010; NCTM, 2000). In order to understand mathematics, students must develop complex networks of knowledge reflecting conscious organization of related mathematical facts and processes (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Klausmeier, Harris, Davis, Schwenn, & Frayer, 1968; Marshall, 1990; Messick, 1984; Skemp, 2006; Webb & Romberg, 1992). Conceptual understanding is therefore defined as “an integrated and functional grasp of mathematical ideas” (National Research Council, 2001, p. 118). Connected knowledge develops over time and experience, and the depth of a person’s understanding grows with the number of
cognitive connections that person constructs among concepts and procedures (Hiebert & Lefevre, 1986).

Knowledge of facts and procedures is instrumental for working mathematically, but symbol manipulations are often required as the chief demonstration of mathematical proficiency (Battista, 1999; Vergnaud, 1997), thus overlooking the importance of the conceptual underpinnings of mathematical processes. A consequence of this oversight is the dominant view that proficiency in mathematics can be explained by procedural fluency, which conveys little about a student’s actual mathematical understanding (Boaler, 2002). Such an approach jeopardizes the potential for students to develop true mathematical proficiency (Battista, 1999).

Assessing student understanding of mathematics requires alternative measurement tools that query conceptual understanding in addition to procedural fluency (Webb & Romberg, 1992). The measurement process used for any assessment should take into account the nature of how people come to know things, how people demonstrate their knowledge, and how observations or test responses are interpreted (Mislevy, Almond, & Lukas, 2003; Mislevy, Steinberg, & Almond, 2003; Mislevy, Steinberg, Almond, Haertel, & Penuel, 2000). Assessments that target understanding should be construct centered (Kane, 2001); that is, they should be developed in response to descriptions of the knowledge, skills, and abilities students are expected to acquire.

**Cognitive Models**

Understanding in a given domain develops with education and experience (Wilson, 2009) and represents a person’s effort “to make sense of many distinct pieces of knowledge” (Wiggins & McTighe, 2005, p. 37). Models of how knowledge develops in a
domain can effectively enhance test design by providing details about the typical sequences in which things are learned, the likely misconceptions held, and the common errors made by students at different stages in their learning. Specifically, cognitive models can provide insight to test developers about how different concepts and skills are cognitively organized, what knowledge should be connected, and the cognitive processes people use when they work through problems (Pellegrino, 1988). Armed with such detailed information, test developers are able to produce carefully designed problem sets that elicit targeted knowledge, provide useful data, and substantiate valid interpretations. Such a scientific approach to test item development can be the foundation for diagnostic tests that not only measure what people know or can do, but also identify knowledge gaps or misconceptions. The diagnostic capacity of such a test is directly related to the fact that the items developed to correspond to a cognitive model permit test administrators to review the meaning of test item responses in terms set out by that model. That is, the cognitive model serves as a guide for assessment design and analysis, which in turn can influence instruction by informing educators more specifically about what students know and can do (Pellegrino et al., 1999).

Learning progressions (Popham, 2011), construct maps (Wilson, 1992, 2009), and learning hierarchies (Gagné, 1968) are three types of cognitive models useful for describing the knowledge people possess, how it is typically learned, and common misconceptions. These types of models can be used to guide the development of assessment items and tasks that measure what people know in a manner that is consistent with how knowledge is acquired (Pellegrino, 1988). Assessments developed in this way can also provide information about what students know in terms of the elements of the
A relatively new type of cognitive model is the cognitive attribute hierarchy, which can be used to model the cognitive processes or skills students need to successfully complete test items (Gierl, Wang, & Zhou, 2008; Leighton, Gierl, & Hunka, 2004). In a cognitive attribute hierarchy, each component of the knowledge, skill, or ability needed to demonstrate proficiency in a domain is represented by an attribute (Leighton et al., 2004). Furthermore, the attributes are arranged hierarchically to depict how people likely acquire domain knowledge, while the dependencies, or cognitive connections, among the attributes are clearly indicated (Leighton et al., 2004).

Overview

The purpose of this study was to: (1) propose a cognitive model depicting how students typically acquire a selected body of content knowledge, (2) develop an assessment guided by the proposed cognitive model, and (3) gather and analyze student test data in order to reflect on the accuracy of that model. Student test responses were analyzed using the Attribute Hierarchy Method (AHM) to provide insight about the validity of the proposed cognitive model. This study responded to the call to apply cognitive diagnostic methods such as the AHM to practical testing situations (Leighton et al., 2004).

Method

Development of the Proposed Cognitive Attribute Hierarchy

The present study investigated the knowledge and skills students need to understand the concept of slope. This study included the development of the Foundational Concepts of Slope Attribute Hierarchy (FCSAH), a model of how students
acquire understanding of selected concepts related to slope. The FCSAH, shown in Figure 1, contains five components of knowledge arranged into a hierarchy reflecting the work of mathematics education researchers who have studied covariation and proportional reasoning.

Drawing from Adamson (2005), the first two attributes in the FCSAH represent the ability to perceive and interpret covariation in different problem contexts. The first attribute describes the ability to detect which quantities are related to one another in a problem (Moritz, 2005), while the second attribute describes the ability to detect the direction of covariation of two quantities (Barr, 1980).

The third, fourth, and fifth attributes represent the ability to interpret the meaning of the slope ratio in terms of a problem’s context variables. This requires students to apply proportional reasoning, which develops only after a person has the ability to conceive of covariation and is able to work with rational numbers (Heller, Post, Behr, & Lesh, 1990). These three attributes describe the ability to interpret slopes whose ratios simplify to different types of fraction values, reflecting the work of Noelting (1980a), who determined that fractions that simplify to whole-numbered values, positive unit fraction values, and positive fractions that are not equivalent to either whole numbers or unit fractions represent different levels of challenge for students.
Figure 1. The Foundational Concepts of Slope Attribute Hierarchy (FCSAH) depicts five attributes associated with understanding the concept of slope. Attribute A1 is defined as the ability to identify covariates from a problem scenario. Attribute A2 is defined as the ability to identify covariates and the direction of their relationship. Attribute A3 is defined as the ability to interpret a slope whose value equals a whole number. Attribute A4 is defined as the ability to interpret a slope whose value simplifies to a positive unit fraction. Attribute A5 is defined as the ability to interpret a slope whose value simplifies to a positive rational number that is neither a whole number nor a unit fraction.

Assessment Development

The study included the development of the Foundational Concepts of Slope Assessment (FCSA), a test designed to evaluate student mastery of the attributes in the FCSAH. Following the process described by Gierl, Leighton, and Hunka (2000), four matrix representations of the FCSAH, shown in Figure 2, were constructed to guide test design and item development.
The first item design targeted knowledge of attribute A1 and required students to identify the quantities that varied together in a problem scenario. The second item design targeted knowledge of attributes A1-A2 and required students to determine the direction of covariation in a problem. The third, fourth, and fifth item designs targeted knowledge of attributes A1-A2-A3, A1-A2-A4, and A1-A2-A5, respectively. These item designs required students to interpret the meaning of a slope ratio in terms of a problem’s context. Items targeting A3 contained slope values that simplified to whole numbers. Items targeting A4 contained slope values that simplified to positive unit fractions. Items targeting A5 contained slope values that simplified to positive rational numbers that were neither whole numbers nor unit fractions.

Four items were developed for each of the five item designs, a variation on the AHM, in which each attribute is represented by a single test item. The resulting FCSA contained 20 multiple-choice items and was designed to provide a quantitative measure of students’ understanding of the attributes modeled in the FCSAH. Each of the items on the FCSA contained four response options, and assessment experts reviewed the items to
ensure their technical quality. The items were written using direct, concise, and grade-appropriate language. Furthermore, the response options were constructed such that each had similar sentence length and parallel language. A balance of mathematical representations was sought in the item stems and answer options. Specifically, the number of items containing graphs in the item stems was equal to the number of items containing graphs in the answer options, and these items were alternated throughout the assessment. The FCSA was administered online in May 2011 to 1,629 middle and high school students studying Pre-algebra, Algebra 1, Geometry, Algebra 2, or any similar course taken prior to Pre-calculus.

Analysis

The methods and results provided in this section illustrate the application of the AHM to actual student test responses. These results are part of a larger study that investigated the development of a theoretical cognitive model and a corresponding assessment that could be used to measure and report student knowledge in terms of the components of the cognitive model. The present study implemented item response theory (IRT) for item and test analyses as well as for student ability estimations. Then the AHM was used to classify students based on their test responses into knowledge states consistent with the FCSAH. Finally, student classifications were analyzed in order to reflect on the accuracy of the proposed cognitive model.

Item and Test Information

Student responses to the items on the FCSA were analyzed using factor analysis and IRT. Factor analyses confirmed unidimensionality. The scree plot is shown in Figure 3. IRT analyses produced an ability estimate for each student. IRT analyses also were
used to describe the items on the FCSA, the FCSA as a test, and the five subtests that each targeted a specific combination of attributes in the FCSAH. Three IRT parameters were determined to describe each item’s discrimination, difficulty, and lower bound, and these parameters determined an item characteristic curve (ICC) for each item. A review of the ICCs, shown in Figure 4, confirmed that many of the items on the FCSA were relatively easy for the participants in this study, while several of the items on the FCSA discriminated well among students of different ability levels. The test information function, shown in Figure 5, also illustrated that the FCSA as a test was relatively easy for the participants, and it provided the most information for students with ability levels from -2.0 to 1.0.

![Figure 3. Scree plot of student responses to the FCSA.](image)
Figure 4. Item characteristic curves for the items on the FCSA.

Figure 5. Test information function and standard error for the FCSA.
The five subtests, each of which targeted knowledge of one specific combination of attributes, were analyzed to determine the test information function for each subtest. A plot of the five subtest information functions is shown in Figure 6. Analyses of these functions revealed that the subtest targeting knowledge of attribute A1, comprised of items numbered 1 through 4, was less informative than the other subtests for the entire population in the study and was most informative for students with ability estimates near the value of -2.0. The subtest targeting knowledge of attributes A1 and A2, comprised of items numbered 5 through 8, was slightly more informative overall than the first subtest and was most informative for students with ability estimates near the value of -0.5. The subtest targeting knowledge of attributes A1, A2, and A3, comprised of items numbered 9 through 12, was the most informative for the entire population in the study and was particularly informative for students with ability estimates near the value of -1.0. The subtest targeting knowledge of attributes A1, A2, and A4, comprised of items numbered 13 through 16, was the second most informative subtest for the entire population in the study and was particularly informative for students with ability estimates near the value of -0.2. The subtest targeting knowledge of attributes A1, A2, and A5, comprised of items numbered 17 through 20, was the third most informative subtest for the entire population in the study and was particularly informative for students with ability estimates near the value of -1.0.
Classification of Students Based on the AHM

Test items target particular content knowledge, and test item responses are determined by the knowledge possessed by examinees (Gierl et al., 2000). If an examinee possesses the knowledge targeted by a test item, then that examinee should answer the test item correctly. One aim of the present study was to classify each student participant into a particular knowledge state with regard to the attributes modeled in the FCSAH. The different levels of knowledge used for classification were the 10 different combinations of attributes consistent with the FCSAH. Each of these 10 attribute combinations was considered to be a different knowledge state, indicating a different level of knowledge of the attributes modeled by the FCSAH. For each knowledge state, an expected response pattern was determined. Each expected response pattern contained

\[\text{Figure 6. Attribute subtest information functions (TIFs).}\]
the responses a hypothetical student would give depending on that student’s level of knowledge. This study assumed that students who possessed an attribute would correctly answer all four items that addressed that attribute. The 10 expected response patterns based on the attribute combinations consistent with the FCSAH are shown in Table 1.

Table 1

<table>
<thead>
<tr>
<th>Knowledge State</th>
<th>Expected Response Pattern</th>
<th>Ability Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>A0</td>
<td>00000000000000000000</td>
<td>-2.92</td>
</tr>
<tr>
<td>A1</td>
<td>11110000000000000000</td>
<td>-2.23</td>
</tr>
<tr>
<td>A12</td>
<td>11111110000000000000</td>
<td>-1.67</td>
</tr>
<tr>
<td>A123</td>
<td>11111111111100000000</td>
<td>-0.95</td>
</tr>
<tr>
<td>A124</td>
<td>11111111000111100000</td>
<td>-1.19</td>
</tr>
<tr>
<td>A125</td>
<td>11111110000000111111</td>
<td>-1.23</td>
</tr>
<tr>
<td>A1234</td>
<td>11111111111111111111</td>
<td>0.14</td>
</tr>
<tr>
<td>A1235</td>
<td>11111111111110001111</td>
<td>-0.21</td>
</tr>
<tr>
<td>A1245</td>
<td>11111111000111111111</td>
<td>-0.42</td>
</tr>
<tr>
<td>A12345</td>
<td>11111111111111111111</td>
<td>1.45</td>
</tr>
</tbody>
</table>

Student responses to the FCSA were examined to determine the level of knowledge possessed by each examinee. Using the AHM, each student’s observed response pattern was compared to all 10 of the expected response patterns. Each comparison followed the method described by Leighton et al. (2004) using the formula shown below, as well as the ability estimates of the hypothetical students associated with each knowledge state consistent with the FCSAH. These comparisons produced estimates of the likelihood that an observed student response pattern matched an expected response pattern. The formula used was:

\[ P_{j \text{expected}}(\theta) = \prod_{k=1}^{K} P_{jk}(\theta) \prod_{m=1}^{M} [1 - P_{jm}(\theta)], \]
where \( j \) corresponds to the expected response pattern to which an actual student’s responses are compared, \( k \) is the number of ones in the difference vector, and \( m \) is the number of negative ones in the difference vector. Each comparison followed this procedure:

First, the student response pattern was subtracted from the expected response pattern to produce a difference vector. Each element of this vector was 1, 0, or -1. An entry of 1 in the difference vector indicated that the hypothetical student with the knowledge implied by the expected response pattern should have answered the item correctly, but the actual student did not—an error. An entry of 0 in the difference vector indicated that the actual student answered the item in the same way as the hypothetical student. An entry of -1 in the difference vector indicated that the student with the knowledge implied by the expected response pattern should have answered the item incorrectly, but the actual student answered the item correctly.

Second, the difference vector was used in the formula with the ability estimate (\( \Theta \)) corresponding to the expected response pattern to yield a likelihood of the actual student having the same ability estimate as that corresponding to the expected response pattern. For each student, the 10 likelihood estimates of how well the student’s observed response pattern matched each of the expected response patterns were calculated.

Third, for each student, the 10 likelihood estimates were summed, and each likelihood estimate was divided by the sum to produce a probability. Each student was assigned to the knowledge state corresponding to the expected response pattern with the highest probability. The comparisons for one student are shown in Table 2.
The student whose comparison is shown in Table 2 was classified in knowledge state A1235 because this category corresponded to the highest likelihood and probability values in Table 2. This student correctly answered all of the items that assessed attributes A1, A2, and A3. Therefore, this student was classified as possessing attributes A1, A2, and A3. This student incorrectly answered items 13 and 20. Item 13 was a relatively easy item that assessed attribute A4. This student would have had to answer item 13 correctly in order to be classified as possessing attribute A4.

Table 2

Likelihood Estimates, Probabilities, and Knowledge State Classification of a Sample Student With Observed Response Pattern 111111111101111110 and Ability Estimate of 0.64

<table>
<thead>
<tr>
<th>Ability Estimate</th>
<th>Expected Response Pattern</th>
<th>$L_j/\text{Expected}(\Theta)$</th>
<th>$P_j/\text{Expected}(\Theta)$</th>
<th>Knowledge State</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.92</td>
<td>00000000000000000000</td>
<td>0.00</td>
<td>0.00</td>
<td>A0</td>
</tr>
<tr>
<td>-2.23</td>
<td>11110000000000000000</td>
<td>0.00</td>
<td>0.00</td>
<td>A1</td>
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<td>11111111000000000000</td>
<td>0.00</td>
<td>0.00</td>
<td>A12</td>
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<tr>
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<td>11111111111110000000</td>
<td>0.03</td>
<td>0.04</td>
<td>A123</td>
</tr>
<tr>
<td>-1.19</td>
<td>11111111100001111000</td>
<td>0.01</td>
<td>0.01</td>
<td>A124</td>
</tr>
<tr>
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<td>11111111100000001111</td>
<td>0.00</td>
<td>0.00</td>
<td>A125</td>
</tr>
<tr>
<td>-0.14</td>
<td>11111111111111110000</td>
<td>0.23</td>
<td>0.34</td>
<td>A1234</td>
</tr>
<tr>
<td>-0.21</td>
<td>11111111111111100011</td>
<td>0.27</td>
<td>0.39</td>
<td>A1235</td>
</tr>
<tr>
<td>-0.42</td>
<td>11111111100001111111</td>
<td>0.12</td>
<td>0.17</td>
<td>A1245</td>
</tr>
<tr>
<td>1.45</td>
<td>11111111111111111111</td>
<td>0.03</td>
<td>0.05</td>
<td>A12345</td>
</tr>
</tbody>
</table>

Note. Each row lists the ability estimate ($\Theta$) and the expected response pattern of the theoretical student possessing the attributes and corresponding expected responses of the particular knowledge state. $L(\Theta)$ is the estimated likelihood that the actual student being compared shared the same ability estimate as the theoretical student. $P(\Theta)$ denotes the probability corresponding to this likelihood.

Since item 13 was missed, this student was classified as not possessing attribute A4. Item 20 was by far the most difficult item that assessed attribute A5. Students who answered this item incorrectly but correctly answered the other three items that assessed
attribute A5 were classified as possessing attribute A5. This is because this item was so much more difficult than the other three items that assessed A5.

All 1,629 student participants were classified into one of the 10 knowledge states that are consistent with the FCSAH. The ability estimates calculated for participants are presented in Table 3, where -2.67 is the minimum value in the “Minimum Ability” column, and 1.45 is the maximum value in the “Maximum Ability” column. Table 3 shows the ability estimate assigned to the expected response vector for each knowledge state, as well as the percent of students classified into each knowledge state. Table 3 also lists the descriptive statistics needed to plot box-and-whisker plots of the abilities of the students assigned to each knowledge state. These box-and-whisker plots are shown in Figure 7, where the relative height of each box is proportional to the percent of students assigned to that knowledge state.

Table 3

*Descriptive Statistics About Knowledge State Classifications*

<table>
<thead>
<tr>
<th>Knowledge State</th>
<th>Ability Estimate</th>
<th>Minimum Ability</th>
<th>Lower Quartile Ability</th>
<th>Median Ability</th>
<th>Upper Quartile Ability</th>
<th>Maximum Ability</th>
<th>Percent of Students</th>
<th>Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>A0</td>
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<td>1</td>
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<td>-1.65</td>
<td>-1.31</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>A12</td>
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<td>-1.81</td>
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<tr>
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<td>88</td>
</tr>
<tr>
<td>A12345</td>
<td>1.45</td>
<td>1.14</td>
<td>1.14</td>
<td>1.14</td>
<td>1.45</td>
<td>1.45</td>
<td>13</td>
<td>100</td>
</tr>
<tr>
<td>All Categories</td>
<td>n/a</td>
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<td>-0.56</td>
<td>0.66</td>
<td>0.10</td>
<td>1.45</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>
Figure 7. Percent of students assigned to each knowledge state by ability estimate.
Students who demonstrated knowledge of none of the attributes in the FCSAH appeared to have different ability levels than the other students. Students who demonstrated knowledge of attributes A1 and A2 appeared to have somewhat similar ability levels, but they appeared to have different ability levels than students who demonstrated knowledge of more attributes. Specifically, students classified as possessing only attribute A1 demonstrated the ability to identify covariates, but did not demonstrate the ability to determine the direction of covariation or to interpret slope ratios. Students classified as possessing attributes A1 and A2 demonstrated the ability to identify covariates and the direction of covariation, but did not demonstrate the ability to interpret slope ratios. Students who demonstrated knowledge of attributes A1 and A2 plus either A3, A4, or A5 appeared to have very similar ability levels, and appeared to have very different ability levels than students who demonstrated knowledge of fewer or more attributes. This group of students demonstrated the ability to identify covariates, the direction of covariation, and the meaning of some slope ratios. Students who demonstrated knowledge of attributes A1 and A2 with any pair of attributes from A3, A4, or A5 appeared to have very similar ability levels, and appeared to have very different ability levels than students who demonstrated knowledge of fewer or more attributes. This group of students demonstrated the ability to identify covariates, the direction of covariation, and the meaning of many of the slope ratios. Students who demonstrated knowledge of all five attributes in the FCSAH appeared to have very different ability levels than students who demonstrated knowledge of fewer attributes.

These results suggest that the ability to interpret slopes whose values simplify
to different types of fractions should not be represented by distinct attributes. Rather, attributes A3, A4, and A5 may represent a single ability.

**Discussion**

Any application of the AHM requires a cognitive model of the content matter being tested. However, very few studies describe the development of a cognitive model that is then used to guide assessment development. More frequently, tests are analyzed after they are developed, and a cognitive model is fit to the items that appear on the test. This study provided an example of how the literature relevant to a content domain can be used to generate a theoretical cognitive model. In this study, the mathematics education literature was consulted to identify the conceptual knowledge students need in order to understand foundational concepts related to slope. The literature revealed that students should understand covariation and be able reason with proportions. Therefore, the hierarchical cognitive model developed in this study included five attributes describing the connections students should be able to make when presented with word problems or graphs representing direct variation situations.

Using the AHM and the proposed cognitive model, students in this study were classified into different knowledge states with regard to their knowledge of selected foundational concepts related to slope. An analysis of these students’ classifications suggested that knowledge of the attributes in the FCSAH, as assessed by the FCSA, develops in a manner that is consistent with the arrangement of the attributes in the FCSAH. These findings support the theory that there are three main levels of understanding of the selected foundational concepts related to slope. First, students demonstrate the ability to identify quantities that are related as covariates. Second,
students demonstrate the ability to identify the direction of covariation in a problem setting. Third, students demonstrate the ability to interpret a slope ratio in terms of a problem’s context variables.

Researchers and practitioners should be aware of different types of cognitive models that are available to represent how concepts and skills are arranged in a person’s cognitive structure. Cognitive models provide means by which specific concepts and skills can be studied to identify essential cognitive connections that students should make, which can then inform curriculum, instruction, and assessment. Educators should be aware of cognitive models or learning progressions that define optimal sequences for specific learning targets in order to foster learning with understanding. Cognitive models that are vetted through empirical tests such as analysis using the AHM can be used to guide curricular planning and the preparation of educational materials. Teachers who are aware of the components of knowledge students need in order to learn with success are better equipped to prepare lessons that are sensitive to prerequisite skills and foster essential cognitive connections.

Few studies illustrate the application of the AHM to actual student test responses. The present study demonstrated that student test responses can be used to confirm the accuracy of the theoretical cognitive model used by the AHM and also to describe student knowledge in terms of the components in the cognitive model. Assessment models like the AHM that can classify students according to what those students already know and what they still need to learn have the potential to powerfully inform instructional decisions.
Limitations and Future Research

This study demonstrated the development of a hierarchical model depicting how a particular concept ideally is learned and how to use that model to inform item and test design. Limitations of this study include the narrow scope of the content selected and the limited types of questions used on the assessment. More studies should be conducted to apply cognitive diagnostic methods such as the AHM to practical assessment situations. Researchers and test developers should continue to investigate the application of assessments and measurement models that effectively classify students according to their demonstrated levels of knowledge. Although these models have been investigated theoretically, more studies are needed to explore how classification models can be used to evaluate the effectiveness of educational assessments taken by actual students. Additional studies should investigate more topics in mathematics and other content areas to develop models of how different concepts and skills are learned and to provide assessment tools whose results can be used to describe student knowledge in terms of the components depicted in the models.

The assessment used in this study contained four test items targeting each of the attributes in the proposed model. Although this design provided students with a reasonable sample of items per attribute, it also introduced a range of ability estimates associated with each attribute. While there appeared to be three levels of knowledge demonstrated by this sample of students, these three levels were not entirely distinct in terms of the ability estimates of the students within these knowledge states. Future studies should explore how researchers might interpret the cases where students’ ability estimates do not clearly lie in distinct knowledge states.
While the present study provided information about how students likely acquire understanding of a particular concept, it was limited in that only a single type of item was used. Similar studies would benefit from using different types of items. While the mathematics education community promotes the notion that mathematical proficiency is supported by both conceptual understanding and procedural fluency (CCSSO/NGA, 2010; National Research Council, 2001; NCTM, 2000), a limitation of the present study was that it only investigated conceptual understanding and specifically avoided test questions in which procedural fluency would either interfere with or enhance student performance. However, procedural skills are also essential in mathematics and represent an area worthy of investigation. Future studies should explore procedural fluency with conceptual understanding to describe how these two types of knowledge interact to enhance student performance with regard to mathematics concepts and skills.

The FCSAH and FCSA were guided by national standards documents (CCSSO/NGA, 2010; NCTM, 2000, 2006, 2009) as well as a variety of mathematics education literature. However, the curriculum standards guiding instruction for students in Kansas describe specific mathematics concepts and skills in terms that may vary considerably from the descriptions used to guide instruction in other states. The grade levels where specific concepts and skills are emphasized in different curricular standards may also vary from state to state. Analysis of test responses from multiple populations of students whose instruction is guided by a wide variety of standards documents may improve the generalizability of the results of any similar study.
Conclusion

As educators implement the Common Core State Standards (CCSS), better instructional tools will be needed. The higher expectations and theoretical foundations of the CCSS will require sophisticated tools to help teachers meet students’ needs. Assessments that diagnose student understandings and misconceptions are useful for guiding instructional decisions. The research community should support educators by exploring models of how content is learned and by producing advanced assessment tools capable of providing instructionally relevant information.
References


